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# The distribution of the Liu-type estimator of the biasing parameter in elliptically contoured models

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**ABSTRACT**
We derive the density function of the stochastic shrinkage parameters of the Liu-type estimator in elliptical models. The correctness of derivation is checked by simulations. A real data application is also provided.

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## 1. Introduction

In regression analysis, researchers often encounter the problem of multicollinearity. In so much econometric work, especially with time-series data, there is often high but not exact multicollinearity. It is well-known that near multicollinearity leads to very large sampling variances. Correlation matrices might have one or more small eigenvalues, leading to large estimates of regression coefficients. Statisticians should choose good experimental designs. If the design matrix has columns nearly collinear, then the least-squares coefficients will be unstable. The ordinary least-squares (OLS) estimator performs poorly in the presence of multicollinearity. Condition number is a measure of the presence of multicollinearity. If $\boldsymbol{Z}'\boldsymbol{Z}$ (where $\boldsymbol{Z}$ denotes the design matrix) is ill conditioned with a large condition number, a biased estimation procedure is desirable. A popular numerical technique to deal with near multicollinearity is that of ridge regression estimator $\widehat{\boldsymbol{\beta}}_R(k) = (\boldsymbol{Z}'\boldsymbol{Z} + k\boldsymbol{I})^{-1}\boldsymbol{Z}'\boldsymbol{y}$ (Hoerl and Kennard, 1970), where $\mathbf{y}$ is the response vector and $k$ is a tuning parameter. To combat near multicollinearity, Liu (1993) proposed an estimator similar in form but different from the ridge estimator of Hoerl and Kennard (1970). The estimator $\widehat{\boldsymbol{\beta}}_L(d) = (\boldsymbol{Z}'\boldsymbol{Z} + \boldsymbol{I})^{-1}(\boldsymbol{Z}'\boldsymbol{y} + d\widehat{\boldsymbol{\beta}}_{OLS})$, where $0 < d < 1$ is a biasing parameter and $\widehat{\boldsymbol{\beta}}_{OLS}$ is the OLS estimator of the regression coefficients, was called the Liu (linear-unified) estimator (Akdeniz and Kacranlar, 1995). See also Kibria and Haq (1999) and Kibria (2012).

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The ridge regression estimator is effective, but as Liu (1993) pointed out, it has the disadvantage that the estimated parameters are complicated non linear functions of the tuning parameter $k$. The advantage of the Liu estimator is that the estimators are linear functions of the biasing parameter $d$. For this reason, this shrinkage estimator has become more popular in recent years. In this note, we use the generalized Liu estimator.

The performance of all operational biased estimators crucially depends upon the distribution of the stochastic shrinkage parameters. Rubio and Firinguetti (2002) derived the density and distribution functions of the stochastic parameters of operational ridge regression estimators. The aim of this note is to derive the density function of the stochastic shrinkage parameters of operational generalized Liu estimators in elliptical models.

Elliptically contoured distributions (ECDs) have a long history as they include multivariate normal and multivariate $t$ distributions as special cases. Some recent developments on them have included elliptically contoured random fields in space and time (Ma, 2010); shrinkage minimax estimation (Tsukuma, 2010); influence diagnostics on the coefficient of variation (Riquelme et al., 2011); near-exact distributions for the likelihood ratio test statistic to test equality of several variance–covariance matrices (Coelho and Marques, 2012); multivariate elliptically contoured stable distributions (Nolan, 2013); and misspecified linear models with elliptically contoured errors (Hu et al., 2015). A comprehensive account of recent developments can be found in the excellent book by Gupta et al. (2013).

The contents of this note are organized as follows. In Section 2, we introduce the general linear model (GLM) and the generalized Liu estimator. In Section 3, estimators of the optimal values for the biasing parameters are determined. We derive their density functions in Section 4. A simulation study and a real data illustration are given in Section 5.

## 2. The model and estimators

The most important model belonging to the class of general linear hypotheses is the GLM. The purpose of GLM is to model the relationship between several independent or predictor variables and a dependent or criterion variable. Consider the GLM

$$\boldsymbol{y} = \boldsymbol{Z}\boldsymbol{\gamma} + \boldsymbol{\epsilon},$$

where $\boldsymbol{y}$ is a $n$-vector of responses, $\boldsymbol{Z}$ is a $n \times p$ non stochastic design matrix with full column rank $p$, $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_p)'$ is a $p$-vector of regression coefficients, and $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)'$ is the $n$-vector of random errors distributed according to an ECD denoted by $\mathcal{E}_n(\mathbf{0}, \sigma^2\boldsymbol{V}, \psi)$. Here, $\sigma \in \mathbb{R}^+$, $\boldsymbol{V}$ is a known matrix belonging to $S(n)$, the set of all positive definite matrices of order $(n \times n)$, and $\boldsymbol{\epsilon}$ has the characteristic function

$$\phi_{\boldsymbol{\epsilon}}(\boldsymbol{t}) = \psi\left(\sigma^2\boldsymbol{t}'\boldsymbol{V}\boldsymbol{t}\right)$$

for some function $\psi : [0, \infty) \to \mathbb{R}$ known as the characteristic generator.

If $\boldsymbol{\epsilon}$ possesses a density, then

$$f(\boldsymbol{\epsilon}) \propto \left|\sigma^2\mathbf{V}\right|^{-\frac{1}{2}} g\left(\frac{1}{\sigma^2}\,\boldsymbol{\epsilon}'\boldsymbol{V}^{-1}\boldsymbol{\epsilon}\right),$$

where $g(\cdot)$ is a non negative function over $\mathbb{R}^+$ such that $f(\cdot)$ is a density function with respect to a $\sigma$-finite measure $\mu$ on $\mathbb{R}^p$. In this case, we write $\boldsymbol{\epsilon} \sim \mathcal{E}_n(\mathbf{0}, \sigma^2\boldsymbol{V}, g)$.

Using the spectral decomposition $\boldsymbol{C} = \boldsymbol{Z}'\boldsymbol{V}^{-1}\boldsymbol{Z} = \boldsymbol{\Gamma}\boldsymbol{\Lambda}\boldsymbol{\Gamma}'$, where $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_p) \in S(p)$ and $\boldsymbol{\Gamma}$ are the matrices of eigenvalues and eigenvectors of $\boldsymbol{C}$, the canonical GLM can be given by

$$y = X\beta + \epsilon,$$

where $X = Z\Gamma$ and $\beta = \Gamma'\gamma$.

The OLS estimator of $\beta$ has the form

$$\widehat{\beta} = \Lambda^{-1}X'V^{-1}y.$$

By least-squares theory in elliptical models (see Arashi et al. (2013) for details), the Liu estimator can be defined as

$$\widetilde{\beta}_d = (\Lambda + I)^{-1}\left(X'V^{-1}y + d\widehat{\beta}\right).$$

Also the generalized Liu estimator can be defined as

$$\widetilde{\beta}_D = (\Lambda + I)^{-1}\left(X'V^{-1}y + D\widehat{\beta}\right),$$

where $D = \text{diag}(d_1, \dots, d_p)$ is a diagonal matrix of the biasing parameters $d_i \in \mathbb{R}$, $i = 1, \dots, p$.

## 3. Optimal biasing parameters

Optimal values for the biasing parameters in the generalized Liu estimator are those $d_i$'s which minimize

$$R\left(\beta; \widetilde{\beta}_D\right) = E\left[\left(\widetilde{\beta}_D - \beta\right)'\left(\widetilde{\beta}_D - \beta\right)\right].$$

Using Equations (3.1) and (3.2) in Arashi et al. (2014), we obtain

$$R\left(\beta; \widetilde{\beta}_D\right) = \sum_{j=1}^{n} \frac{\left[\sigma_\epsilon^2\left(\lambda_j + d_j\right)^2 + \lambda_j(1 - d_j)^2\beta_j^2\right]}{\lambda_j(1 + \lambda_j)^2},$$

where $\sigma_\epsilon^2 = -2\psi'(0)\sigma^2$ and $\psi'(\cdot)$ is the first derivative of $\psi(\cdot)$. Thus, $R(\beta; \widetilde{\beta}_D)$ is minimized at

$$d_{i(\text{opt})} = \frac{\lambda_i\left(\beta_i^2 - \sigma_\epsilon^2\right)}{\lambda_i\beta_i^2 + \sigma_\epsilon^2}.$$

An estimate of $d_i$ can be obtained by substituting unbiased estimates of $\sigma^2$ and $\beta_i$ into $d_{i(\text{opt})}$, yielding

$$\widehat{d_i} = \frac{\lambda_i\left(\widehat{\beta}_i^2 - \widehat{\sigma}_\epsilon^2\right)}{\lambda_i\widehat{\beta}_i^2 + \widehat{\sigma}_\epsilon^2}.$$

We call the estimator, $\widehat{d_i}$, the OLS-based minimum risk estimator for $d_{i(\text{opt})}$. Thus, we obtain a feasible (operational) version of the generalized Liu estimator. It is noteworthy that the optimal value $d_{i(\text{opt})}$ providing the minimum risk for the Liu type estimator is independent of the elliptical assumption.

In the next section, the density function of $\widehat{d_i}$ is obtained under the assumption of ellipticity in the GLM.

## 4. Distribution of the stochastic shrinkage parameter of the Liu estimator

The main result of this note is the following theorem.

**Theorem 1.** *Under the assumptions of the GLM, the density function of the shrinkage estimator* $\widehat{d}_i$ *is*

$$f_{\widehat{d}_i}(x) = \frac{\nu^{\frac{\nu}{2}}\left(\frac{1+\lambda_i}{1-x}-1\right)^{-\frac{1}{2}}\frac{1+\lambda_i}{(1-x)^2}}{B\left(\frac{1}{2},\frac{\nu}{2}\right)\left(\nu-1+\frac{1+\lambda_i}{1-x}\right)^{\frac{1}{2}(\nu+1)}} \cdot\sum_{j=1}^{\infty}\mathcal{C}(j)\left[\frac{\frac{1+\lambda_i}{1-x}-1}{2\left(\nu-1+\frac{1+\lambda_i}{1-x}\right)}\right]^j\frac{\Gamma\left(\frac{1}{2}(\nu+1)+j\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(j+1\right)\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(\nu+1)\right)}, \quad -\lambda_i < x < 1,$$

*where*

$$\mathcal{C}(j) = \int \Delta_t^j \exp\left(-\frac{1}{2}\Delta_t\right)\mathcal{W}(t)\mathrm{d}t,$$

*where* $\Delta_t = t\theta_i$, $\theta_i = \lambda_i\beta_i^2/\sigma^2$, $\nu = n-p$, *and* $\mathcal{W}(t)$ *is the weighting function of elliptical models.*

**Proof.** Let

$$u = \frac{\nu}{\sigma_\epsilon^2}\widehat{\sigma}_\epsilon^2, \qquad \widehat{\sigma}_\epsilon^2 = -2\psi'(0)\widehat{\sigma}^2, \qquad \delta_i = \frac{\lambda_i}{\sigma_\epsilon^2}\widehat{\beta}_i^2, \qquad w = \frac{\nu\delta_i}{u}.$$

Using the results in Arashi et al. (2013) and Theorem 3.1 of Akdeniz and Ozturk (2005), we have $\delta_i|t \sim \chi_1^2(\Delta_t)$ (non central Chi-square distribution with non centrality parameter $\Delta_t$) independent of $u \sim \chi_\nu^2$. Consequently, $w|t \sim F_{1,\nu,\Delta_t}$ (non central $F$ distribution with non centrality parameter $\Delta_t$). Hence,

$$\begin{aligned} f(w) &= \int \mathcal{W}(t) f_{1,\nu,\Delta_t}(w|t)\mathrm{d}t \\ &= \frac{\nu^{\frac{\nu}{2}}w^{-\frac{1}{2}}}{B\left(\frac{1}{2},\frac{\nu}{2}\right)(\nu+w)^{\frac{1}{2}(\nu+1)}}\sum_{j=1}^{\infty}\left[\frac{w}{2(\nu+w)}\right]^j\frac{\Gamma\left(\frac{1}{2}(\nu+1)+j\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(j+1\right)\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(\nu+1)\right)} \\ &\quad \cdot\int \Delta_t^j\exp\left(-\frac{1}{2}\Delta_t\right)\mathcal{W}(t)\mathrm{d}t. \end{aligned}$$

Since

$$\widehat{d}_i = \frac{\nu\delta_i - u\lambda_i}{\nu\delta_i + u} = 1 - \frac{1+\lambda_i}{1+w},$$

using the inverse transformation $w = \frac{1+\lambda_i}{1-\widehat{d}_i} - 1$, the density of $\widehat{d}_i$ can be obtained as

$$f_{\widehat{d}_i}(x) = \frac{\nu^{\frac{\nu}{2}}\left(\frac{1+\lambda_i}{1-x}-1\right)^{-\frac{1}{2}}\frac{1+\lambda_i}{(1-x)^2}}{B\left(\frac{1}{2},\frac{\nu}{2}\right)\left(\nu-1+\frac{1+\lambda_i}{1-x}\right)^{\frac{1}{2}(\nu+1)}} \cdot\sum_{j=1}^{\infty}\mathcal{C}(j)\left[\frac{\frac{1+\lambda_i}{1-x}-1}{2\left(\nu-1+\frac{1+\lambda_i}{1-x}\right)}\right]^j\frac{\Gamma\left(\frac{1}{2}(\nu+1)+j\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(j+1\right)\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(\nu+1)\right)}, \quad -\lambda_i < x < 1.$$

The proof is complete. □

The weighting function $\mathcal{W}(\cdot)$ is not necessarily of positive value on its domain. Thus, the distribution of $\widehat{d_i}$ should not be mistakenly interpreted as a scale mixture of normal distributions.

Suppose the distribution of errors in the GLM is multivariate $t$ specified by the joint density function

$$f(\boldsymbol{\epsilon}) = \frac{\Gamma\left(\frac{\gamma_o+n}{2}\right)\left|\sigma^2\boldsymbol{V}\right|^{-\frac{1}{2}}}{(\pi\gamma_o)^{\frac{n}{2}}\Gamma\left(\frac{\gamma_o}{2}\right)}\left\{1+\frac{1}{\gamma_o\sigma^2}\boldsymbol{\epsilon}'\boldsymbol{V}^{-1}\boldsymbol{\epsilon}\right\}^{-\frac{1}{2}(\gamma_o+n)},$$

where $\gamma_o$ is the degrees of freedom. We write $\boldsymbol{\epsilon} \sim t_n(\boldsymbol{0}, \sigma^2\boldsymbol{V}, \gamma_o)$. Saleh et al. (2014) document the details of statistical inference for the multivariate $t$ distribution.

According to Arashi et al. (2014), the weighting function $\mathcal{W}(\cdot)$ is given by

$$\mathcal{W}(t) = \frac{1}{\Gamma\left(\frac{\gamma_o}{2}\right)}\left(\frac{\gamma_o t}{2}\right)^{\frac{\gamma_o}{2}} e^{-\frac{\gamma_o t}{2}} t^{-1}, \quad t \in \mathbb{R}^+. \tag{1}$$

Substituting (1) into Theorem 1, we obtain the density function of $\widehat{d_i}$ as

$$f_{\widehat{d_i}}(x) = \left(\frac{\gamma_o}{\theta_i+1}\right)^{\frac{\gamma_o}{2}} \frac{\nu^{\frac{\nu}{2}}\left(\frac{1+\lambda_i}{1-x}-1\right)^{-\frac{1}{2}}\frac{1+\lambda_i}{(1-x)^2}}{\Gamma\left(\frac{\gamma_o}{2}\right)B\left(\frac{1}{2},\frac{\nu}{2}\right)\left(\nu-1+\frac{1+\lambda_i}{1-x}\right)^{\frac{1}{2}(\nu+1)}} \cdot\sum_{j=1}^{\infty}\left[\frac{\theta_i\left(\frac{1+\lambda_i}{1-x}-1\right)}{(\theta_i+1)\left(\nu-1+\frac{1+\lambda_i}{1-x}\right)}\right]^j \frac{\Gamma\left(\frac{1}{2}(\nu+1)+j\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(j+\frac{\gamma_o}{2}\right)}{\Gamma\left(j+1\right)\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(\nu+1)\right)}, \quad -\lambda_i < x < 1. \tag{2}$$

Figure 1 shows the possible shapes of (2) for varying $\nu$, $\lambda$, $\gamma_o$, and $\theta$. The density becomes scaled down with increasing values of $\lambda$ but becomes increasingly bimodal with decreasing values of $\lambda$. The density changes from being bathtub shaped to unimodal as $\nu$ increases from 1 to 10. The density changes from being monotonically decreasing to unimodal as $\gamma_o$ increases from 0.1 to 10. The density appears bathtub shaped for the chosen values for $\theta$.

## 5. Numerical demonstrations

### 5.1. Simulation study

When the errors of the GLM are normal, Akdeniz and Ozturk (2005) showed that the simulated distribution of $\widehat{d_i}$, $i = 1, \ldots, p$ approximated to the theoretical version. Here, we consider the case that the errors of the GLM are $t$ distributed. We follow the approach of Akdeniz and Ozturk (2005). We generated an artificial dataset in which $\boldsymbol{y} \sim t_{25}(\boldsymbol{Z\gamma}, 4\boldsymbol{I}, \gamma_o)$, where $\gamma_o = 5$, $\boldsymbol{V} = \boldsymbol{I}$, $\boldsymbol{\gamma} = (1\ 1\ 1)'$, $\boldsymbol{Z} = [\boldsymbol{1}_{25}\ \boldsymbol{z}_1\ \boldsymbol{z}_2]$, where the design points $(z_{1j}, z_{2j})$, $j = 1, \ldots, 25$ were selected according to Figure 2 giving $\boldsymbol{\Lambda} = \mathrm{diag}(288, 25, 8)$.

We followed the Monte Carlo algorithm in Efron (1982) to draw a sample $\boldsymbol{\epsilon}^*$ by putting mass 1/25 at each component of the residual vector $\widehat{\boldsymbol{\epsilon}} = \boldsymbol{y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}$. We then set $\boldsymbol{y}^* = \boldsymbol{X}\widehat{\boldsymbol{\beta}} + \boldsymbol{\epsilon}^*$. The estimated shrinkage parameters are given componentwise as

$$\widehat{d_i^*} = \frac{\lambda_i\left(\left\{\widehat{\beta}_i^*\right\}^2 - \left\{\widehat{\sigma}_{\boldsymbol{\epsilon}}^*\right\}^2\right)}{\lambda_i\left\{\widehat{\beta}_i^*\right\}^2 + \left\{\widehat{\sigma}_{\boldsymbol{\epsilon}}^*\right\}^2}, \quad i = 1, 2, 3,$$

where $\{\widehat{\sigma}_{\boldsymbol{\epsilon}}^*\}^2 = \gamma_o D^2/\nu(\gamma_o - 2)$ and $D^2 = (\boldsymbol{y}^* - \boldsymbol{X}\widehat{\boldsymbol{\beta}}^*)'(\boldsymbol{y}^* - \boldsymbol{X}\widehat{\boldsymbol{\beta}}^*)$.

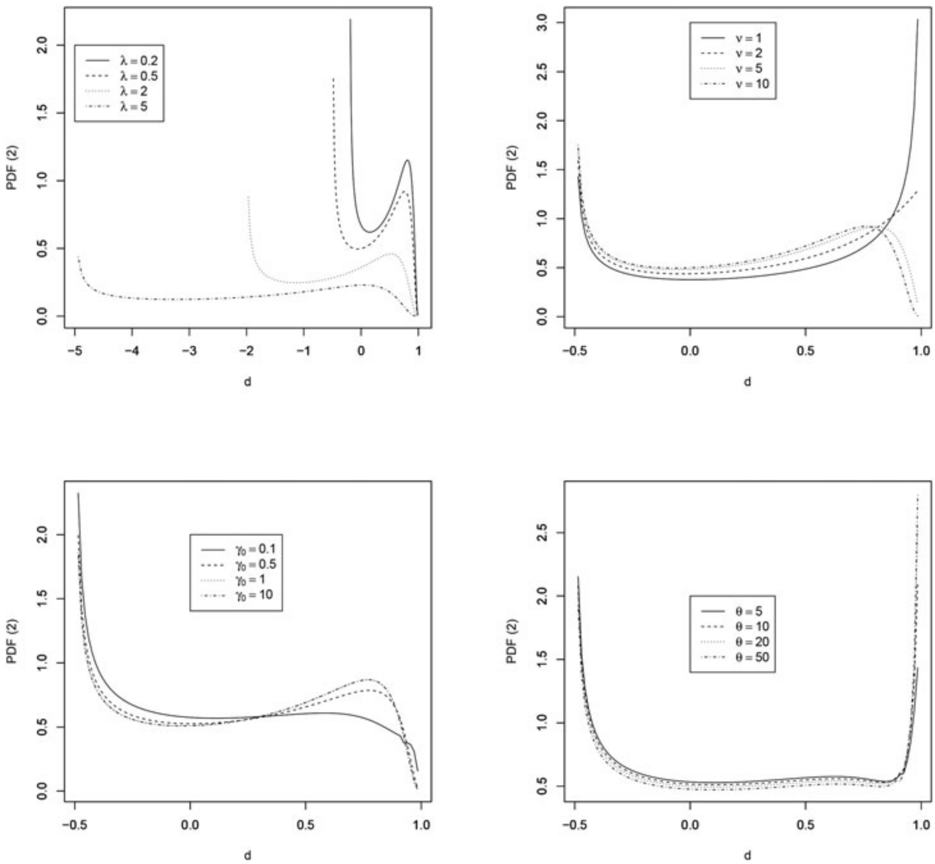


**Figure 1.** The density function (2) for selected values of $\nu$, $\lambda$, $\gamma_o$, and $\theta$.

For $B = 1000$ repetitions, Figure 3 shows the histograms of the simulated values

$$\widehat{d}^*_{iB} = \frac{\lambda_i\left(\left\{\widehat{\beta}^*_{iB}\right\}^2 - \left\{\widehat{\sigma}^*_{\epsilon}\right\}^2\right)}{\lambda_i\left\{\widehat{\beta}^*_{iB}\right\}^2 + \left\{\widehat{\sigma}^*_{\epsilon}\right\}^2}.$$

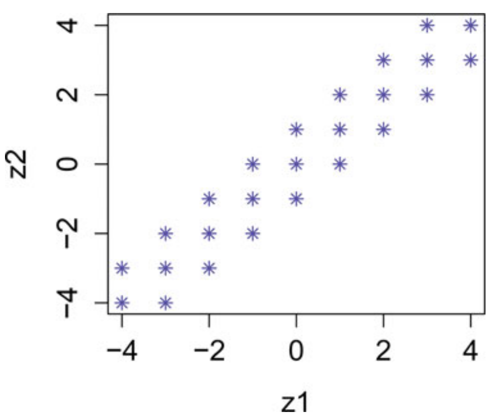


**Figure 2.** Design points $(z_1, z_2)$.

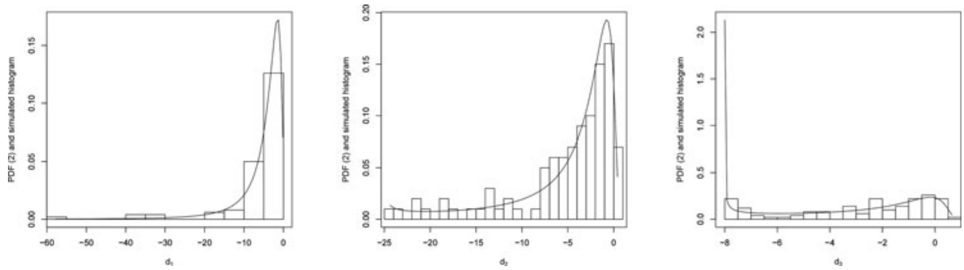

Figure 3. Simulated histograms and the density function (2) for $\gamma_o = 5$.

Also plotted in the figure are the exact density functions given by (2). The closeness of the densities to the histograms suggests that the derivations in the earlier sections are correct.

### 5.2. Real data application

Here, we calculate a feasible version of the generalized Liu estimator based on a real dataset. We consider the dataset on Portland cement originally due to Woods et al. (1932). A listing of the data is given by

$$\boldsymbol{Z} = \begin{bmatrix} 7 & 26 & 6 & 60 \\ 1 & 29 & 15 & 52 \\ 11 & 56 & 8 & 20 \\ 11 & 31 & 8 & 47 \\ 7 & 52 & 6 & 33 \\ 11 & 55 & 9 & 22 \\ 3 & 71 & 17 & 6 \\ 1 & 31 & 22 & 44 \\ 2 & 54 & 18 & 22 \\ 21 & 47 & 4 & 26 \\ 1 & 40 & 23 & 34 \\ 11 & 66 & 9 & 12 \\ 10 & 68 & 8 & 12 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} 78.5 \\ 74.3 \\ 104.3 \\ 87.6 \\ 95.9 \\ 109.2 \\ 102.7 \\ 72.5 \\ 93.1 \\ 115.9 \\ 83.8 \\ 113.3 \\ 109.4 \end{bmatrix}.$$

The data come from an experimental investigation of the heat evolved during the setting and hardening of Portland cements of varied composition and the dependence of this heat on the percentages of four compounds in the clinkers from which the cement was produced. The dependent variable $\boldsymbol{y}$ is defined as heat evolved in calories per gram of cement. The independent variables are amounts of the following four compounds: tricalcium aluminate ($\boldsymbol{z}_1$), tricalcium silicate ($\boldsymbol{z}_2$), tetracalcium alumino ferrite ($\boldsymbol{z}_3$), and dicalcium silicate ($\boldsymbol{z}_4$).

This data have been previously analyzed by Kaciranlar et al. (1999) and Arashi et al. (2015). The latter demonstrated that the multivariate normal distribution is not a correct distribution for the error term in the GLM and suggested a heavier tail distribution. Here, we take the error distribution as multivariate $t$.

The $13 \times 4$ matrix $\boldsymbol{Z}$ has eigenvalues $\lambda_1 = 211.336941$, $\lambda_2 = 77.235610$, $\lambda_3 = 28.4596570$, and $\lambda_4 = 10.266734$. The condition number of $\boldsymbol{Z}$ is equal to $\kappa = \lambda_1/\lambda_4 = 20.584530$ and also the variance inflation factors are VIF1 $= 38.46154$, VIF2 $= 250$, VIF3 $= 47.61905$, and VIF4 $= 250$, where VIF$i = 1/(1 - R_i^2)$, and $R_i^2$ denotes the coefficient of determination of the

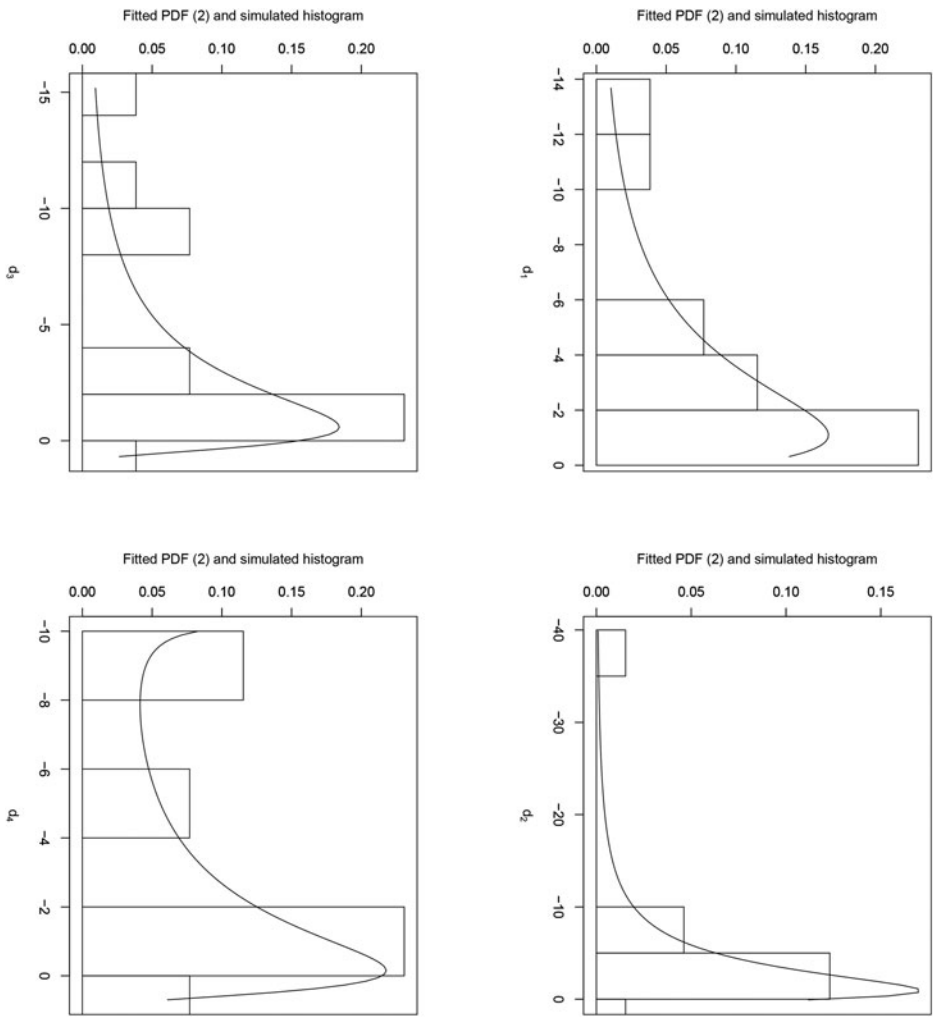


**Figure 4.** Simulated histograms and the fitted density function (2) for $\gamma_o = 5$.

regression equation. These numbers indicate a moderate multicollinearity problem among the regressors.

Figure 4 plots the fitted densities of $d_i$ in (2) versus simulated densities. The simulated densities were obtained as follows: refit the GLM with one row of $\boldsymbol{Z}$ removed at a time; each fit gives an estimate for $d_i$; the simulate density is the histogram of these estimates. The fits shown in the figure appear reasonable.

## 6. Conclusions

We have derived the density function of the stochastic shrinkage parameters proposed by Liu (1993) and Akdeniz and Kacranlar (1995). The optimal shrinkage parameters are functions of the unknown model parameters $\beta_i$ and $\sigma_\epsilon^2$ estimated from the data. The density function of the shrinkage parameters for the generalized Liu-type estimator depends on $\nu$, $\theta_i$, $\lambda_i$, and $\mathcal{W}(t)$ (the weighting function of elliptical models). The correctness of the derivation of the density has been checked by simulation with the errors in the GLM assumed to come from a multivariate $t$ distribution.

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