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Asymptotic Behavior of the Zero Solutions to Generalized Pipe and Rotating Shaft Equations

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Abstract

A non-autonomous partial differential equation describing the dynamics of a uniform pipe and a system describing the dynamics of a rotating shaft are considered.Sufficient conditions for the global asymptotic stability of the zero solution of the boundary value problem for the differential equation and the system under consideration are established by using the Lyapunov function technique.

Key Words: Stability, Lyapunov function

1. Introduction

In this paper we are concerned with the global asymptotic stability of the following equation, for $x \in (0, L), t \in \mathbb{R}^+$

$$w_{tt} + aw_{xxxx} + bw_{txxxx} + cw_t + \alpha(t)w_{xx} + l(x)\beta(t)w_{xx} + \gamma(t)w_{xt} = 0, \tag{1}$$

under the boundary conditions

$$w(0,t) = w(L,t) = w_x(0,t) = w_x(L,t) = 0, t \in \mathbb{R}^+ \tag{2}$$

or

$$w(0,t) = w(L,t) = w_{xx}(0,t) = w_{xx}(L,t) = 0, t \in \mathbb{R}^+ \tag{3}$$

where a, b, c are given positive constants, and $\alpha(t), \beta(t), \gamma(t), l(x)$ are positive bounded functions and the following system

$$v_{tt} + \alpha_1 v_{xxxx} - a(t)w_t - b(t)v - c(t)w + \beta_1 v_{txxxx} + \gamma v_t - d(t)w = 0 \tag{4}$$

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$$w_{tt} + \alpha_2 w_{xxxx} - a(t)v_t - b(t)w + c(t)v + \beta_2 w_{txxxx} + \gamma w_t + d(t)v = 0 \tag{5}$$

under the boundary conditions

$$\begin{aligned} v(0,t) &= v(L,t) = w(0,t) = w(L,t) \\ &= v_x(0,t) = v_x(L,t) = w_x(0,t) = w_x(L,t) = 0, \ \ t \in \mathbb{R}^+ \end{aligned} \tag{6}$$

or

$$\begin{aligned} v(0,t) &= v(L,t) = w(0,t) = w(L,t) \\ &= v_{xx}(0,t) = v_{xx}(L,t) = w_{xx}(0,t) = w_{xx}(L,t) = 0 \ , t \in \mathbb{R} \end{aligned} \tag{7}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ and γ are given positive constants and $a(t), b(t), c(t), d(t)$ are positive bounded functions.

In the article of Plaut [4] the following equation

$$\begin{aligned} 0 &= EIw_{xxxx} + d_i EIw_{txxxx} + d_e w_t + MU^2 w_{xx} + (L-x)M\dot{U}w_{xx} \\ &\quad +2MUw_{xt} + (M+m)w_{tt} \end{aligned} \tag{8}$$

under the boundary conditions (2) (or (3)) and the following system

$$EI_1 v_{xxxx} + mv_{tt} - 2m\Omega w_t - m\Omega^2 v - m\dot{\Omega} w + d_i EI_1 v_{txxxx} + d_e m(v_t - \Omega w) = 0 \tag{9}$$

$$EI_2 w_{xxxx} + mw_{tt} - 2m\Omega v_t - m\Omega^2 w + m\dot{\Omega} v + d_i EI_2 w_{txxxx} + d_e m(w_t + \Omega v) = 0 \tag{10}$$

under the boundary conditions (6) (or (7)) are considered. Equation (8) describes the dynamics of a uniform pipe of length L,mass per unit length m,bending stiffness EI,conveying a fluid of mass per unit length M and time varying velocity $U(t)$ in the positive x direction, d_i and d_e are internal and external damping coefficients respectively. Equations (9) and (10) describe the dynamics of a rotating shaft with time-varying angular velocity $\Omega(t)$.The quantities L, m, d_i, d_e are the same as in equation (8).EI_1 and EI_2 are bending stiffnesses. In [4] it is proven that the zero solution of the problem (8),(2) (or (8),(3)) is stable with respect to the norm $(\int_0^L (w_{xx}^2 + w_t^2)dx)^{\frac{1}{2}}$ and the zero solution of the problem (9),(10), (6) (or (9),(10),(7)) is stable with respect to the norm $\int_0^L (v_{xx}^2 + w_{xx}^2 + v_t^2 + w_t^2)dx)^{\frac{1}{2}}$ under certain conditions.

Our main aim is to find sufficient conditions guaranteeing the global asymptotic stability of the zero solution of (1),(2)(or(1),(3)) and (4),(5),(6) (or (4),(5),(7)) and to generalize the previous results of PLAUT [4] to equations with arbitrary time-dependent coefficients. The Lyapunov function technique is used to obtain our results. The method used in this paper is also applicable to nonlinear systems. KALANTAROV and KURT [2] examined the global asymptotic stability of a class of similar equations with nonlinear dissipative terms.

The state vector is denoted $\mathbf{u}$ and the equilibrium state is $\mathbf{u} = 0$. The state space $\mathcal{U}$ contains the elements $\mathbf{u}$ which satisfy the boundary conditions and appropriate smoothness conditions. An initial state at $t = 0$ is $\mathbf{u}_0$ and the ensuing motion is $\mathbf{u}(t, \mathbf{u}_0)$. A specific norm $\|.\|$ is defined on $\mathcal{U}$. The extended Lyapunov's direct method requires the construction of a functional $\mathcal{W}$, which is defined in the state space $\mathcal{U}$ having the following properties (DYM [1], MOVCHAN [3], ZUBOV [6]). The subsequent three properties of the function $\mathcal{W}$ represent a sufficient condition for **stability** of the equilibrium state $\mathbf{u} = 0$.

1. There exists a $c_1 > 0$ so that for every $\mathbf{u} \in \mathcal{U}$,$\mathcal{W}(\mathbf{u}) \leq c_1 \|\mathbf{u}\|^2$.

2. There exists a $c_2 > 0$ so that for every $\mathbf{u} \in \mathcal{U}$,$\mathcal{W}(\mathbf{u}) \geq c_2 \|\mathbf{u}\|^2$.

3. $\frac{d}{dt}\mathcal{W}(\mathbf{u}(t, \mathbf{u}_0)) \leq 0$.

The zero solution is called **globally asymptotically stable** if the zero solution is stable and all solutions tend to zero, in the appropriate sense, as $t \to \infty$.

We shall use the following notations throughout:

$$\|w\| = (\int_0^L w^2(x)dx)^{\frac{1}{2}},\ (w, v) = \int_L^0 w(x)v(x)dx.$$

We also use the Wirtinger inequality [5]

$$\int_0^L w^2(x)dx \leq \lambda \int_0^L w_x^2(x)dx \tag{11}$$

where $\lambda = \frac{L^2}{\pi^2}$ if both ends are pinned, $\lambda = \frac{L^2}{4\pi^2}$ if both ends are clamped. After the integration of the equality $(ww_x)_x = w_x^2 + ww_{xx}$with respect to x and using Cauchy and

Wirtinger inequalities respectively we obtain at once

$$\int_0^L w_x^2(x)dx \leq \lambda \int_0^L w_{xx}^2(x)dx. \tag{12}$$

Using (11)and (12), it is not difficult to see that

$$\int_0^L w^2(x)dx \leq \lambda^2 \int_0^L w_x^2(x)dx. \tag{13}$$

2. Global Asymptotic Stability

Theorem 2.1. *Suppose that the following conditions are satisfied:*

i) a, b, c *are given positive constants,*

ii) $\alpha(.), \beta(.)$*are positive functions from* $C^1[0,\infty)$,*satisfying the conditions*

$$\alpha(t) + \beta(t)L + \beta(t)L_1\sqrt{\lambda} + \frac{\gamma^2(t)}{2} \leq a_0 \tag{14}$$

$$\left|\alpha^{'}(t)\right| + \left|\beta^{'}(t)\right| L + \beta^2(t)L_1 \leq \eta_0(\frac{a}{\lambda} - a_0) \tag{15}$$

where a_0 *is an arbitrary positive number which satisfies*

$$a_0 < \frac{a}{\lambda}, \text{ and} \, \eta_0 = \min\left\{1, \frac{a - a_0\lambda}{|\lambda^2 - b|}, \frac{2(\frac{b}{\lambda^2} + c) - L_1}{3}\right\}$$

iii) $\gamma(.)$*is a positive function from* $C[0,\infty)$

iv) $l(.) \in C^1[0, L]$ *and*

$$0 \leq l(x) \leq L, \tag{16}$$

$$\left|l^{'}(x)\right| \leq L_1, \forall x \in [0, L], \tag{17}$$

where L_1 is a positive number satisfying

$$L_1 < 2(\frac{b}{\lambda^2} + c). \tag{18}$$

Then the zero solution of (1),(2) (or (1), (3)) is globally asymptotically stable with respect to the norm

$$(\int_0^L (w_{xx}^2 + w_t^2)dx)^{\frac{1}{2}}.$$

Moreover for every solution of equation (1) satisfying the boundary conditions (2) or (3) the following estimate holds:

$$\|w_{xx}\|^2 + \|w_t\|^2 \leq K_1 e^{-\delta t} \tag{19}$$

where K_1 and δ are positive parameters.

Proof. Suppose that $w(x,t)$ is a solution of equation (1) satisfying the boundary conditions (2(or (3)) and η is a parameter to be specified later. Multiplying the equation (1) by $w_t + \eta w$ and using the boundary conditions(2) (or (3)) we obtain:

$$\frac{d}{dt}E_\eta(w, w_t) + H_\eta(w, w_t) = 0 \tag{20}$$

where

$$\begin{aligned} E_\eta(w, w_t) \quad &= \quad \eta(w_t, w) + \eta\tfrac{b}{2}\|w_{xx}\|^2 + \eta\tfrac{c}{2}\|w\|^2 + \tfrac{1}{2}\|w_t\|^2 + \tfrac{a}{2}\|w_{xx}\|^2 \\ &\quad -\tfrac{1}{2}\alpha(t)\|w_x\|^2 - \tfrac{1}{2}\beta(t)\int_0^L l(x)w_x^2 dx \end{aligned} \tag{21}$$

and

$$\begin{aligned} H_\eta(w, w_t) \quad &= \quad -\eta\alpha(t)\|w_x\|^2 + \eta a\|w_{xx}\|^2 - \eta\|w_t\|^2 \\ &\quad -\eta\beta(t)\int_0^L l(x)w_x^2 dx - \eta\beta(t)\int_0^L l'(x)w_x w dx - \eta\gamma(t)(w_x, w_t) \\ &\quad +\tfrac{1}{2}\alpha'(t)\|w_x\|^2 + \tfrac{1}{2}\beta'(t)\int_0^L l(x)w_x^2 dx \\ &\quad -\beta(t)\int_0^L l'(x)w_x w_t dx + c\|w_t\|^2 + b\|w_{txx}\|^2. \end{aligned} \tag{22}$$

By using (13) we obtain the following inequality:

$$\eta\left|(w,w_t)\right| \le \eta\lambda\left\|w_t\right\|\left\|w_{xx}\right\| \le \eta\frac{\lambda^2}{2}\left\|w_{xx}\right\|^2 + \frac{\eta}{2}\left\|w_t\right\|^2. \tag{23}$$

Using (23), (12), and (13) in (21) we get:

$$E_\eta(w,w_t) \ge \frac{1}{2}[a-\eta(\lambda^2-b)-(\alpha(t)+\beta(t)L)\lambda]\left\|w_{xx}\right\|^2 + \frac{1}{2}(1-\eta)\left\|w_t\right\|^2. \tag{24}$$

If $\lambda^2 - b < 0$, from (14) and for $\eta < 1$ we have:

$$E_\eta(w,w_t) \ge k_0(\left\|w_{xx}\right\|^2 + \left\|w_t\right\|^2) \tag{25}$$

for a suitable constant k_0. If $\lambda^2 - b > 0$, from (14) and for

$$\eta < \min\left\{1, \frac{a-a_0\lambda}{\lambda^2-b}\right\} \tag{26}$$

we have

$$E_\eta(w,w_t) \ge k_1(\left\|w_{xx}\right\|^2 + \left\|w_t\right\|^2) \tag{27}$$

for a suitable constant k_1. Using (23) in (21) we obtain:

$$\begin{aligned} E_\eta(w,w_t) &\le \eta\tfrac{\lambda^2}{2}\left\|w_{xx}\right\|^2 + \tfrac{\eta}{2}\left\|w_t\right\|^2 + \eta\tfrac{b}{2}\left\|w_{xx}\right\|^2 + \eta c\tfrac{\lambda^2}{2}\left\|w_{xx}\right\|^2 \\ &+ \tfrac{1}{2}\left\|w_t\right\|^2 + \tfrac{a}{2}\left\|w_{xx}\right\|^2 + \tfrac{\lambda}{2}(\alpha(t)+\beta(t)L)\left\|w_{xx}\right\|^2 \\ &\le \tfrac{1}{2}(\eta\lambda^2+\eta b+\eta c\lambda^2+a+a_0\lambda)\left\|w_{xx}\right\|^2 + \tfrac{1}{2}(1+\eta)\left\|w_t\right\|^2. \end{aligned} \tag{28}$$

For

$$k_2 = \frac{1}{2}max\left\{\eta\lambda^2+\eta b+\eta c\lambda^2+a+a_0\lambda, 1+\eta\right\} \tag{29}$$

we obtain from (28) that:

$$E_\eta(w,w_t) \le k_2(\left\|w_{xx}\right\|^2 + \left\|w_t\right\|^2). \tag{30}$$

Using (12) and (17) we obtain the following inequalities:

$$\gamma(t)\left|(w_x,w_t)\right| \le \frac{\gamma^2(t)}{2}\left\|w_x\right\|^2 + \frac{1}{2}\left\|w_t\right\|^2 \tag{31}$$

$$\beta(t)\int_0^L l'(x)w_x w dx \le \beta(t)L_1\sqrt{\lambda}\left\|w_x\right\|^2 \tag{32}$$

$$\beta(t)\int_0^L l'(x)w_x w_t dx \le \frac{\beta^2(t)}{2}L_1\left\|w_x\right\|^2+\frac{L_1}{2}\left\|w_t\right\|^2. \tag{33}$$

By using (12),(13),(16),(31),(32) and (33) in (22) we obtain that the following inequality holds:

$$\begin{aligned} H_\eta(w,w_t) \quad & \ge [\eta(\tfrac{a}{\lambda}-\alpha(t)-\beta(t)L-\beta(t)L_1\sqrt{\lambda} \\ & -\tfrac{\gamma^2(t)}{2})-\tfrac{1}{2}(\left|\alpha'(t)\right|+\left|\beta'(t)\right|L+\beta^2(t)L_1)]\left\|w_x\right\|^2 \\ & +(\tfrac{b}{\lambda^2}+c-\tfrac{L_1}{2}-\tfrac{3}{2}\eta)\left\|w_t\right\|^2 \end{aligned} \tag{34}$$

If we use the conditions (14)-(16), we obtain $H_\eta(w,w_t)\ge 0$ and $\frac{d}{dt}E_\eta(w,w_t)\le 0$ for

$$\frac{\eta_0}{2}\le\eta\le\eta_0 \tag{35}$$

where $\eta_0=\min\left\{1,\frac{a-a_0\lambda}{|\lambda^2-b|},\frac{2(\frac{b}{\lambda^2}+c)-L_1}{3}\right\}$.So we obtained that $E_\eta(w,w_t)$ is a Lyapunov functional for the problem (1), (2) (and (1),(3)). Thus the zero solution of (1), (2) (and (1),(3)) is stable with respect to the norm $(\int_0^L(w_{xx}^2+w_t^2)dx)^{\frac{1}{2}}$. Let δ be a positive number; then we obtain from (20) that:

$$\begin{aligned} \tfrac{d}{dt}E_\eta(w,w_t)+\delta E_\eta(w,w_t) \quad = \quad & \delta\eta(w_t,w)+\delta\eta\tfrac{b}{2}\left\|w_{xx}\right\|^2 \\ & +\delta\eta\tfrac{c}{2}\left\|w\right\|^2+\tfrac{\delta}{2}\left\|w_t\right\|^2+\tfrac{a\delta}{2}\left\|w_{xx}\right\|^2 \\ & -\tfrac{\delta}{2}\alpha(t)\left\|w_x\right\|^2-\tfrac{\delta}{2}\beta(t)\int_0^L l(x)w_x^2dx \\ & +\eta\alpha(t)\left\|w_x\right\|^2-\eta a\left\|w_{xx}\right\|^2+\eta\left\|w_t\right\|^2 \\ & +\eta\beta(t)\int_0^L l(x)w_x^2dx+\eta\beta(t)\int_0^L l'(x)w_x w dx \\ & +\eta\gamma(t)(w_x,w_t)-\tfrac{1}{2}\alpha'(t)\left\|w_x\right\|^2 \\ & -\tfrac{1}{2}\beta'(t)\int_0^L l(x)w_x^2dx+\beta(t)\int_0^L l'(x)w_x w_t dx \\ & -c\left\|w_t\right\|^2-b\left\|w_{txx}\right\|^2. \end{aligned} \tag{36}$$

By using (11),(12),(13) and (14) we obtain from (36) that:

$$\begin{aligned}\frac{d}{dt}E_\eta(w,w_t)+\delta E_\eta(w,w_t) \leq\ & (\tfrac{\delta(a+a_0\lambda)}{2}+\delta\eta\tfrac{\lambda^2}{2}+\delta\eta\tfrac{b}{2}+\delta\eta c\tfrac{\lambda^2}{2} \\ & +\eta\lambda(\alpha(t)+\beta(t)L+\beta(t)L_1\sqrt{\lambda} \\ & +\tfrac{\gamma^2(t)}{2})-\eta a+\tfrac{\lambda}{2}(\left|\alpha^{'}(t)\right|+\left|\beta^{'}(t)\right|L \\ & +\beta^2(t)L_1)\left\|w_{xx}\right\|^2 \\ & +(\tfrac{\delta\eta}{2}+\tfrac{\delta}{2}+\tfrac{3}{2}\eta+\tfrac{L_1}{2}-c-\tfrac{b}{\lambda^2})\left\|w_t\right\|^2\end{aligned} \tag{37}$$

From the conditions (14)-(18), and for η chosen in (35), we obtain from (37):

$$\begin{aligned}\frac{d}{dt}E_\eta(w,w_t)+\delta E_\eta(w,w_t) \leq\ & (\tfrac{\delta(a+a_0\lambda)}{2}+\delta\eta\tfrac{\lambda^2}{2}+\delta\eta\tfrac{b}{2}+\delta\eta c\tfrac{\lambda^2}{2} \\ & -(\eta-\tfrac{\eta_0}{2})(a-a_0\lambda))\left\|w_{xx}\right\|^2 \\ & +(\tfrac{\delta\eta}{2}+\tfrac{\delta}{2}+\tfrac{3}{2}\eta+\tfrac{L_1}{2}-c-\tfrac{b}{\lambda^2})\left\|w_t\right\|^2.\end{aligned} \tag{38}$$

Choosing $\delta>0$ sufficiently small in (38) we obtain:

$$\frac{d}{dt}E_\eta(w,w_t)+\delta E_\eta(w,w_t)\leq 0. \tag{39}$$

It follows from the last inequality that:

$$E_\eta(w,w_t)\leq E_\eta(w(x,0),w_t(x,0))e^{-\delta t}. \tag{40}$$

Thus (25) (or (27)) and (40) imply the required inequality (19). □

Theorem 2.2. *Suppose*

i) $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma$ *are given positive constants,*

ii) $a(.),b(.),c(.),d(.)$ *are positive bounded functions of* $C^1[0,\infty)$ *satisfying the following conditions:*

$$b(t)+\frac{a^2(t)}{2}\leq\alpha_0 \tag{41}$$

$$\left|b^{'}(t)\right|+c(t)+d(t)\leq\eta_0(\frac{\alpha}{\lambda^2}-\alpha_0) \tag{42}$$

$$a(t)+\frac{c(t)}{2}+\frac{d(t)}{2}\leq\gamma_0 \tag{43}$$

where $\alpha_0,\beta_0,\gamma_0$ *are any positive numbers satisfying* $\alpha_0<\frac{\alpha}{\lambda^2}=\frac{\min\{\alpha_1,\alpha_2\}}{\lambda^2}$, $\beta_0=\min\{\beta_1,\beta_2\}$, $\gamma_0<\gamma$, $\eta_0=\min\left\{1,\frac{\alpha_1-\alpha_0\lambda^2}{|\lambda^2+\lambda^2\gamma-\beta_1|},\frac{\alpha_2-\alpha_0\lambda^2}{|\lambda^2+\lambda^2\gamma-\beta_2|},\frac{2(\gamma-\gamma_0+\frac{\beta_0}{\lambda^2})}{3}\right\}$. *Then the zero solution of (4),(5),(6) (or (4),(5), (7)) is globally asymptotically stable with respect to the norm* $(\int\limits_0^L(v_{xx}^2+w_{xx}^2+v_t^2+w_t^2)dx)^{\frac{1}{2}}$. *Moreover for every solution of the system (4),(5) satisfying the boundary conditions (6) (or(7)) the following estimate holds:*

$$\|v_{xx}\|^2+\|w_{xx}\|^2+\|v_t\|^2+\|w_t\|^2\leq K_2e^{-\delta t} \tag{44}$$

where K_2 *and* δ *are positive parameters.*

Proof. Let $v(x,t)$ be a solution of the equation (4) and $w(x,t)$ be a solution of equation (5) satisfying the boundary conditions (6) (or (7)) and η be a positive parameter which will be chosen. Multiplying equation (4) by $v_t+\eta v$ and equation (5) by $w_t+\eta w$, using the boundary conditions (6) (or (7)) and adding the obtained equalities, we get the following equality:

$$\begin{aligned}0 \;=\;& \tfrac{d}{dt}[\tfrac{1}{2}\|v_t\|^2+\tfrac{1}{2}\|w_t\|^2+\tfrac{\alpha_1}{2}\|v_{xx}\|^2+\tfrac{\alpha_2}{2}\|w_{xx}\|^2\\ &-\tfrac{b(t)}{2}\|v\|^2-\tfrac{b(t)}{2}\|w\|^2+\eta(v_t,v)+\eta(w_t,w)+\eta\tfrac{\beta_1}{2}\|v_{xx}\|^2\\ &+\eta\tfrac{\beta_2}{2}\|w_{xx}\|^2+\eta\tfrac{\gamma}{2}\|v\|^2+\eta\tfrac{\gamma}{2}\|w\|^2]-2a(t)(v_t,w_t)\\ &+\tfrac{b'(t)}{2}\|v\|^2+\tfrac{b'(t)}{2}\|w\|^2-c(t)(v_t,w)+c(t)(w_t,v)\\ &+\beta_1\|v_{txx}\|^2+\beta_2\|w_{txx}\|^2+\gamma\|v_t\|^2+\gamma\|w_t\|^2-d(t)(v_t,w)\\ &+d(t)(w_t,v)+\eta\alpha_1\|v_{xx}\|^2+\eta\alpha_2\|w_{xx}\|^2-\eta\|v_t\|^2\\ &-\eta\|w_t\|^2-\eta a(t)(w_t,v)-\eta a(t)(v_t,w)-\eta b(t)\|v\|^2-\eta b(t)\|w\|^2.\end{aligned} \tag{45}$$

Let

$$\begin{aligned}\Phi_\eta(t) \;&=\tfrac{1}{2}\|v_t\|^2+\tfrac{1}{2}\|w_t\|^2+\tfrac{\alpha_1}{2}\|v_{xx}\|^2+\tfrac{\alpha_2}{2}\|w_{xx}\|^2\\ &-\tfrac{b(t)}{2}\|v\|^2-\tfrac{b(t)}{2}\|w\|^2+\eta(v_t,v)+\eta(w_t,w)\\ &+\eta\tfrac{\beta_1}{2}\|v_{xx}\|^2+\eta\tfrac{\beta_2}{2}\|w_{xx}\|^2+\eta\tfrac{\gamma}{2}\|v\|^2+\eta\tfrac{\gamma}{2}\|w\|^2.\end{aligned} \tag{46}$$

Thanks to the inequality (13) we have:

$$\eta \left|(w_t, w)\right| \leq \frac{\eta}{2} \left\|w_t\right\|^2 + \frac{\eta}{2}\lambda^2 \left\|w_{xx}\right\|^2 \tag{47}$$

$$\eta \left|(v_t, v)\right| \leq \frac{\eta}{2} \left\|v_t\right\|^2 + \frac{\eta}{2}\lambda^2 \left\|v_{xx}\right\|^2 . \tag{48}$$

Using (47), (48) and (13) in (46) we get:

$$\begin{aligned} \Phi_\eta(t) \quad \geq \quad & \tfrac{1}{2}(\alpha_1 - b(t)\lambda^2 - \eta(\lambda^2 + \lambda^2\gamma - \beta_1)) \left\|v_{xx}\right\|^2 + \tfrac{1}{2}(\alpha_2 - b(t)\lambda^2 \\ & -\eta(\lambda^2 + \lambda^2\gamma - \beta_2)) \left\|w_{xx}\right\|^2 + \tfrac{1}{2}(1-\eta) \left\|v_t\right\|^2 + \tfrac{1}{2}(1-\eta) \left\|w_t\right\|^2 . \end{aligned} \tag{49}$$

If $\lambda^2 + \lambda^2\gamma - \beta_1 < 0$ and $\lambda^2 + \lambda^2\gamma - \beta_2 < 0$ from (41) and for $\eta < 1$ we obtain

$$\Phi_\eta(t) \geq A_0(\left\|v_{xx}\right\|^2 + \left\|w_{xx}\right\|^2 + \left\|v_t\right\|^2 + \left\|w_t\right\|^2) \tag{50}$$

for a suitable constant A_0. If $\lambda^2 + \lambda^2\gamma - \beta_1 < 0$ and $\lambda^2 + \lambda^2\gamma - \beta_2 > 0$ for

$$\eta < \min\left\{1, \frac{\alpha_2 - \alpha_0\lambda^2}{\lambda^2 + \lambda^2\gamma - \beta_2}\right\} \tag{51}$$

we have

$$\Phi_\eta(t) \geq A_1(\left\|v_{xx}\right\|^2 + \left\|w_{xx}\right\|^2 + \left\|v_t\right\|^2 + \left\|w_t\right\|^2) \tag{52}$$

for a suitable constant A_1. If $\lambda^2 + \lambda^2\gamma - \beta_1 > 0$ and $\lambda^2 + \lambda^2\gamma - \beta_2 < 0$ for

$$\eta < \min\left\{1, \frac{\alpha_1 - \alpha_0\lambda^2}{\lambda^2 + \lambda^2\gamma - \beta_1}\right\} \tag{53}$$

we have

$$\Phi_\eta(t) \geq A_2(\left\|v_{xx}\right\|^2 + \left\|w_{xx}\right\|^2 + \left\|v_t\right\|^2 + \left\|w_t\right\|^2) \tag{54}$$

for a suitable constant A_2. If $\lambda^2 + \lambda^2\gamma - \beta_1 > 0$ and $\lambda^2 + \lambda^2\gamma - \beta_2 > 0$ for

$$\eta < \min\left\{1, \frac{\alpha_1 - \alpha_0\lambda^2}{\lambda^2 + \lambda^2\gamma - \beta_1}, \frac{\alpha_2 - \alpha_0\lambda^2}{\lambda^2 + \lambda^2\gamma - \beta_2}\right\} \tag{55}$$

we get

$$\Phi_\eta(t) \geq A_3(\left\|v_{xx}\right\|^2 + \left\|w_{xx}\right\|^2 + \left\|v_t\right\|^2 + \left\|w_t\right\|^2) \tag{56}$$

for a suitable constant A_3. From (46) we get the following inequality:

$$\begin{aligned}\Phi_\eta(t) \leq\ & \tfrac{1}{2}\|v_t\|^2 + \tfrac{1}{2}\|w_t\|^2 + \tfrac{\alpha_1}{2}\|v_{xx}\|^2 + \tfrac{\alpha_2}{2}\|w_{xx}\|^2 \\ & + \tfrac{b(t)}{2}\|v\|^2 + \tfrac{b(t)}{2}\|w\|^2 + \eta|(v_t, v)| + \eta|(w_t, w)| \\ & + \eta\tfrac{\beta_1}{2}\|v_{xx}\|^2 + \eta\tfrac{\beta_2}{2}\|w_{xx}\|^2 + \eta\tfrac{\gamma}{2}\|v\|^2 + \eta\tfrac{\gamma}{2}\|w\|^2.\end{aligned} \tag{57}$$

Using (41) and (13) we get:

$$\begin{aligned}\Phi_\eta(t) \leq\ & \tfrac{1}{2}(\alpha_1 + \alpha_0\lambda^2 + \eta\lambda^2 + \eta\beta_1 + \eta\gamma\lambda^2)\|v_{xx}\|^2 + \tfrac{1}{2}(\alpha_2 + \alpha_0\lambda^2 \\ & + \eta\lambda^2 + \eta\beta_2 + \eta\gamma\lambda^2)\|w_{xx}\|^2 + \tfrac{1}{2}(1+\eta)\|v_t\|^2 + \tfrac{1}{2}(1+\eta)\|w_t\|^2.\end{aligned} \tag{58}$$

For

$$A_4 = \frac{1}{2}\max\{\alpha_1 + \alpha_0\lambda^2 + \eta\lambda^2 + \eta\beta_1 + \eta\gamma\lambda^2, \alpha_2 + \alpha_0\lambda^2 + \eta\lambda^2 + \eta\beta_2 + \eta\gamma\lambda^2, 1+\eta\} \tag{59}$$

we obtain from the last inequality:

$$\Phi_\eta(t) \leq A_4(\|v_{xx}\|^2 + \|w_{xx}\|^2 + \|v_t\|^2 + \|w_t\|^2). \tag{60}$$

In (45) let:

$$\begin{aligned}B_\eta(t) =\ & -2a(t)(v_t, w_t) + \tfrac{b'(t)}{2}\|v\|^2 + \tfrac{b'(t)}{2}\|w\|^2 - c(t)(v_t, w) \\ & + c(t)(w_t, v) + \beta_1\|v_{txx}\|^2 + \beta_2\|w_{txx}\|^2 + \gamma\|v_t\|^2 + \gamma\|w_t\|^2 \\ & - d(t)(v_t, w) + d(t)(w_t, v) + \eta\alpha_1\|v_{xx}\|^2 + \eta\alpha_2\|w_{xx}\|^2 \\ & - \eta\|v_t\|^2 - \eta\|w_t\|^2 - \eta a(t)(w_t, v) - \eta a(t)(v_t, w) \\ & - \eta b(t)\|v\|^2 - \eta b(t)\|w\|^2.\end{aligned} \tag{61}$$

Using (12) we obtain the following inequalities:

$$2a(t)|(v_t, w_t)| \leq a(t)\|v_t\|^2 + a(t)\|w_t\|^2 \tag{62}$$

$$|(v_t, w)| \leq \frac{1}{2}\|v_t\|^2 + \frac{\lambda}{2}\|w_x\|^2 \tag{63}$$

$$|(w_t, v)| \leq \frac{1}{2}\|w_t\|^2 + \frac{\lambda}{2}\|v_x\|^2 \tag{64}$$

$$a(t)|(v_t, w)| \leq \frac{1}{2}\|v_t\|^2 + \frac{a^2(t)}{2}\lambda\|w_x\|^2 \tag{65}$$

$$a(t)\left|(w_t,v)\right| \leq \frac{1}{2}\left\|w_t\right\|^2 + \frac{a^2(t)}{2}\lambda\left\|v_x\right\|^2 \tag{66}$$

Due to (62)-(66) we obtain from (61):

$$\begin{aligned} B_\eta(t) \quad & \geq (\tfrac{\eta\alpha_1}{\lambda} - \eta b(t)\lambda - \eta\tfrac{a^2(t)}{2}\lambda - \tfrac{\left|b'(t)\right|}{2}\lambda \\ & -\tfrac{c(t)}{2}\lambda - \tfrac{d(t)}{2}\lambda)\left\|v_x\right\|^2 \\ & +(\tfrac{\eta\alpha_2}{\lambda} - \eta b(t)\lambda - \eta\tfrac{a^2(t)}{2}\lambda - \tfrac{\left|b'(t)\right|}{2}\lambda \\ & -\tfrac{c(t)}{2}\lambda - \tfrac{d(t)}{2}\lambda)\left\|w_x\right\|^2 \\ & +(\gamma + \tfrac{\beta_1}{\lambda^2} - a(t) - \tfrac{c(t)}{2} - \tfrac{d(t)}{2} - \tfrac{3}{2}\eta)\left\|v_t\right\|^2 \\ & +(\gamma + \tfrac{\beta_2}{\lambda^2} - a(t) - \tfrac{c(t)}{2} - \tfrac{d(t)}{2} - \tfrac{3}{2}\eta)\left\|w_t\right\|^2 . \end{aligned} \tag{67}$$

Using the conditions (41)-(43) for

$$\frac{\eta_0}{2} \leq \eta \leq \eta_0 \tag{68}$$

where $\beta_0 = \min\{\beta_1, \beta_2\}$ and $\eta_0 = \min\{1, \frac{\alpha_1 - \alpha_0\lambda^2}{|\lambda^2+\lambda^2\gamma-\beta_1|}, \frac{\alpha_2-\alpha_0\lambda^2}{|\lambda^2+\lambda^2\gamma-\beta_2|}, \frac{2(\gamma-\gamma_0+\frac{\beta_0}{\lambda^2})}{3}\}$ we obtain $B_\eta(t) \geq 0$. So the zero solution of (4), (5), (6) (or (4), (5), (7)) is stable. Let $\delta > 0$, we get from (45):

$$\begin{aligned} \tfrac{d}{dt}\Phi_\eta(t) + \delta\Phi_\eta(t) \quad = \quad & 2a(t)(v_t,w_t) - \tfrac{b'(t)}{2}\left\|v\right\|^2 - \tfrac{b'(t)}{2}\left\|w\right\|^2 - \beta_1\left\|v_{txx}\right\|^2 \\ & -\beta_2\left\|w_{txx}\right\|^2 + c(t)(v_t,w) - c(t)(w_t,v) - \gamma\left\|v_t\right\|^2 \\ & -\gamma\left\|w_t\right\|^2 + d(t)(v_t,w) - d(t)(w_t,v) - \eta\alpha_1\left\|v_{xx}\right\|^2 \\ & -\eta\alpha_2\left\|w_{xx}\right\|^2 + \eta\left\|v_t\right\|^2 + \eta\left\|w_t\right\|^2 + \eta a(t)(w_t,v) \\ & +\eta a(t)(v_t,w) + \eta b(t)\left\|v\right\|^2 + \eta b(t)\left\|w\right\|^2 + \tfrac{\delta}{2}\left\|v_t\right\|^2 \\ & +\tfrac{\delta}{2}\left\|w_t\right\|^2 + \delta\tfrac{\alpha_1}{2}\left\|v_{xx}\right\|^2 + \delta\tfrac{\alpha_2}{2}\left\|w_{xx}\right\|^2 - \delta\tfrac{b(t)}{2}\left\|v\right\|^2 \\ & -\delta\tfrac{b(t)}{2}\left\|w\right\|^2 + \delta\eta(v_t,v) + \delta\eta(w_t,w) + \delta\eta\tfrac{\beta_1}{2}\left\|v_{xx}\right\|^2 \\ & +\delta\eta\tfrac{\beta_2}{2}\left\|w_{xx}\right\|^2 + \delta\eta\tfrac{\gamma}{2}\left\|v\right\|^2 + \delta\eta\tfrac{\gamma}{2}\left\|w\right\|^2 . \end{aligned} \tag{69}$$

By using the inequalities (62)-(66) we obtain from the above equality:

$$\begin{aligned}\frac{d}{dt}\Phi_\eta(t)+\delta\Phi_\eta(t) \leq\ & [\delta\tfrac{\alpha_1}{2}+\delta\eta\tfrac{\beta_1}{2}+\delta\eta\gamma\tfrac{\lambda^2}{2}+\delta\eta\tfrac{\lambda^2}{2}-\eta\alpha_1\\ &+\eta\lambda^2(b(t)+\tfrac{a^2(t)}{2})+\tfrac{\lambda^2}{2}(\left|b'(t)\right|+c(t)\\ &+d(t))]\left\|v_{xx}\right\|^2+[\delta\tfrac{\alpha_2}{2}+\delta\eta\tfrac{\beta_2}{2}+\delta\eta\gamma\tfrac{\lambda^2}{2}+\delta\eta\tfrac{\lambda^2}{2}\\ &-\eta\alpha_2+\eta\lambda^2(b(t)+\tfrac{a^2(t)}{2})+\tfrac{\lambda^2}{2}(\left|b'(t)\right|\\ &+c(t)+d(t))]\left\|w_{xx}\right\|^2+(\tfrac{\delta}{2}+\tfrac{\delta\eta}{2}+a(t)+\tfrac{c(t)}{2}\\ &+\tfrac{d(t)}{2}+\tfrac{3}{2}\eta-\gamma-\tfrac{\beta_1}{\lambda^2})\left\|v_t\right\|^2+(\tfrac{\delta}{2}+\tfrac{\delta\eta}{2}+a(t)\\ &+\tfrac{c(t)}{2}+\tfrac{d(t)}{2}+\tfrac{3}{2}\eta-\gamma-\tfrac{\beta_2}{\lambda^2})\left\|w_t\right\|^2\end{aligned}\tag{70}$$

or

$$\begin{aligned}\frac{d}{dt}\Phi_\eta(t)+\delta\Phi_\eta(t) \leq\ & [\delta\tfrac{\alpha_1}{2}+\delta\eta\tfrac{\beta_1}{2}+\delta\eta\gamma\tfrac{\lambda^2}{2}+\delta\eta\tfrac{\lambda^2}{2}\\ &-(\eta-\tfrac{\eta_0}{2})(\alpha-\alpha_0\lambda^2)]\left\|v_{xx}\right\|^2\\ &+[\delta\tfrac{\alpha_2}{2}+\delta\eta\tfrac{\beta_2}{2}+\delta\eta\gamma\tfrac{\lambda^2}{2}+\delta\eta\tfrac{\lambda^2}{2}\\ &-(\eta-\tfrac{\eta_0}{2})(\alpha-\alpha_0\lambda^2)]\left\|w_{xx}\right\|^2\\ &+(\tfrac{\delta}{2}+\tfrac{\delta\eta}{2}+a(t)+\tfrac{c(t)}{2}+\tfrac{d(t)}{2}+\tfrac{3}{2}\eta-\gamma-\tfrac{\beta_1}{\lambda^2})\left\|v_t\right\|^2\\ &+(\tfrac{\delta}{2}+\tfrac{\delta\eta}{2}+a(t)+\tfrac{c(t)}{2}+\tfrac{d(t)}{2}+\tfrac{3}{2}\eta-\gamma-\tfrac{\beta_2}{\lambda^2})\left\|w_t\right\|^2.\end{aligned}\tag{71}$$

For sufficiently small $\delta > 0$ we obtain:

$$\frac{d}{dt}\Phi_\eta(t)+\delta\Phi_\eta(t)\leq 0. \tag{72}$$

From (72) we have

$$\Phi_\eta(t)\leq\Phi_\eta(0)e^{-\delta t}. \tag{73}$$

Thus we obtain the required inequality (44). □

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